A new tile with surround number 2

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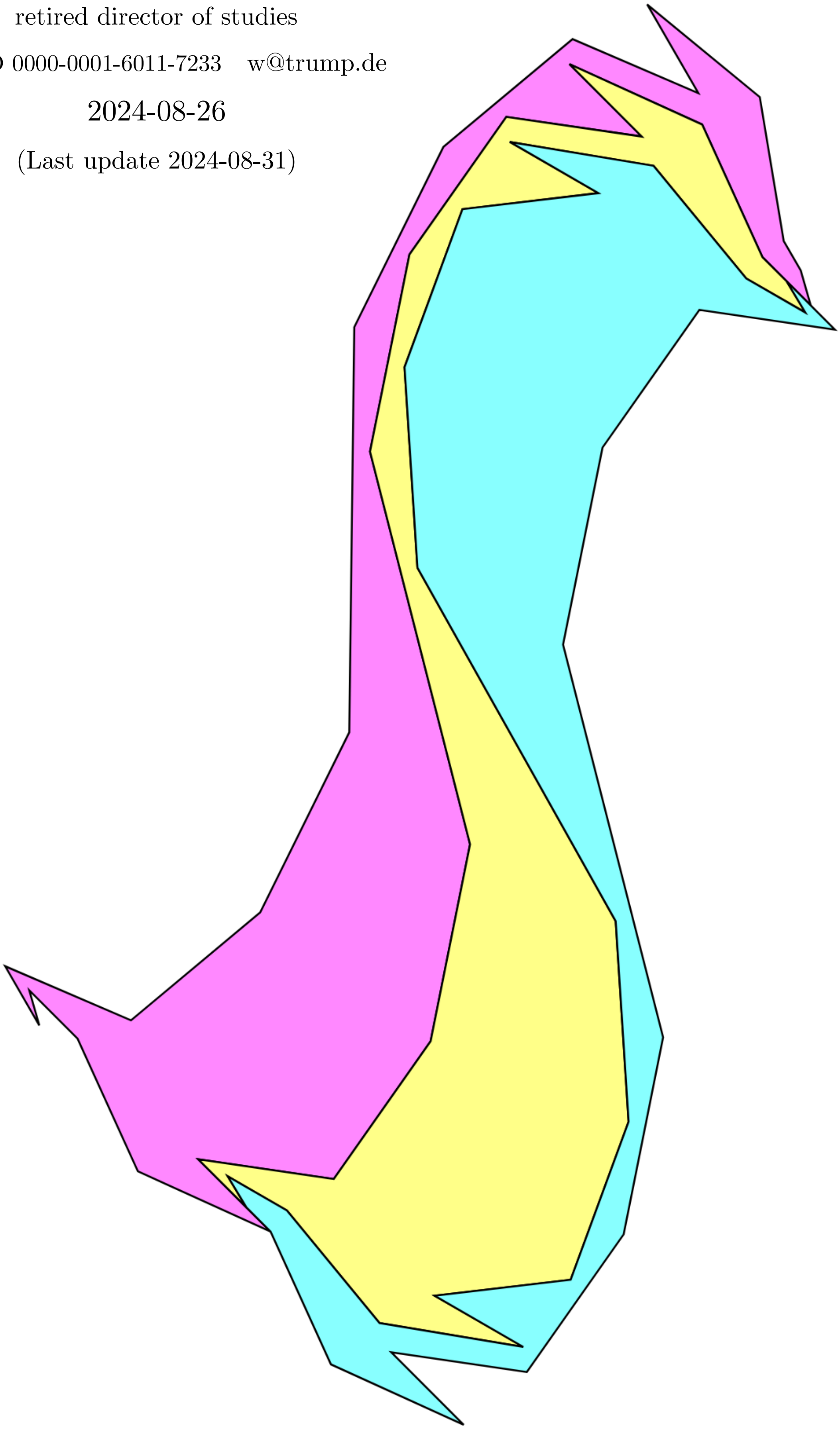

Abstract

A new tile (polygon) with surround number 2 is presented. The tile can be completely surrounded by two copies of itself. Moreover the tile admits periodic and non-periodic tilings of the plane. This answers an open question printed in the paper *Your Friendly Neighborhood Voderberg Tile* [3] and in *The Tiling Book* [4].

1 Definitions

In this paper the definitions given in *The Tiling Book* of Colin Adams [4] are used.
For example a tile T is (*completely*) *surrounded* by a finite set of other tiles if the whole tiling S is a topological disk and T has no point in common with the boundary of S.

2 The tile of Heinz Voderberg

A Voderberg tile [1] can be *enveloped* (*incompletely surrounded*) by two copies of itself. But the tile is not completely surrounded by the two copies, because the two corners A and B of the inner yellow tile touch the boundary of the whole tiling.

Fig. 1 to 4 were constructed with GeoGebra [7].

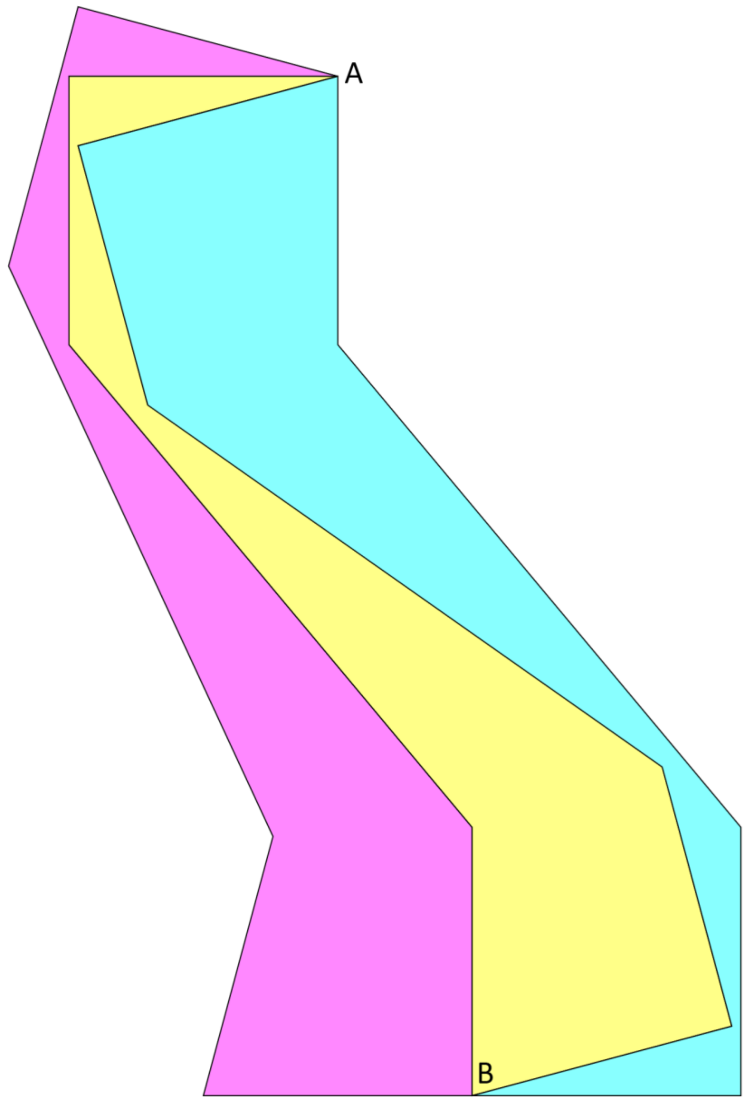

Fig. 1. A Voderberg tile (yellow) is enveloped by two copies of itself. (drawn by the author)

3 The tile of Casey Mann

In 2002 Casey Mann published the first tile with surround number 2. He modified a Voderberg tile by "strategic placement of some hooks and catches". [2]

The shown green tile has the same orientation as the yellow Voderberg tile in Fig. 1. Rotate the green tile by 15° (180°) in order to obtain the same Orientation as the violet (blue) Voderberg tile. See [5] for a modification with larger hooks and catches.

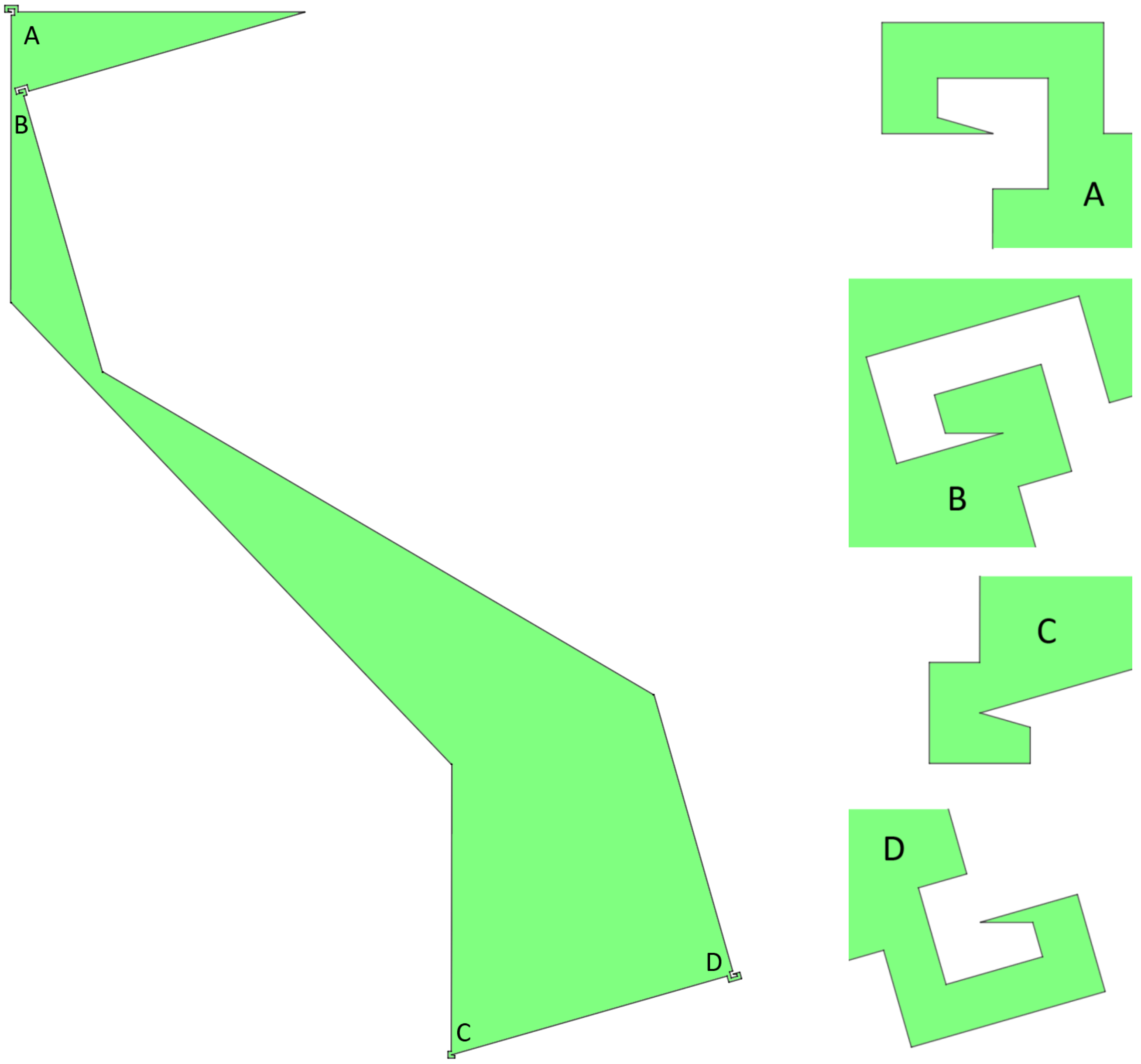

Fig. 2. Casey Mann's tile with microscopic details (drawn by the author)

4 The new tile

Here a polygon with 29 edges (29-gon) is presented (Fig. 3), read in chapter 6 how it can be constructed. This tile can be completely surrounded by two copies of itself (Fig. 4) and it can tile the plane periodically and non-periodically (see chapter 5).
Many different variants of the tile are possible including 25-gons (Fig. 10). The yellow tile in Fig. 4 is achieved by rotating the violet tile by 15° counterclockwise. A rotation of the yellow tile by 180° leads to the blue tile.

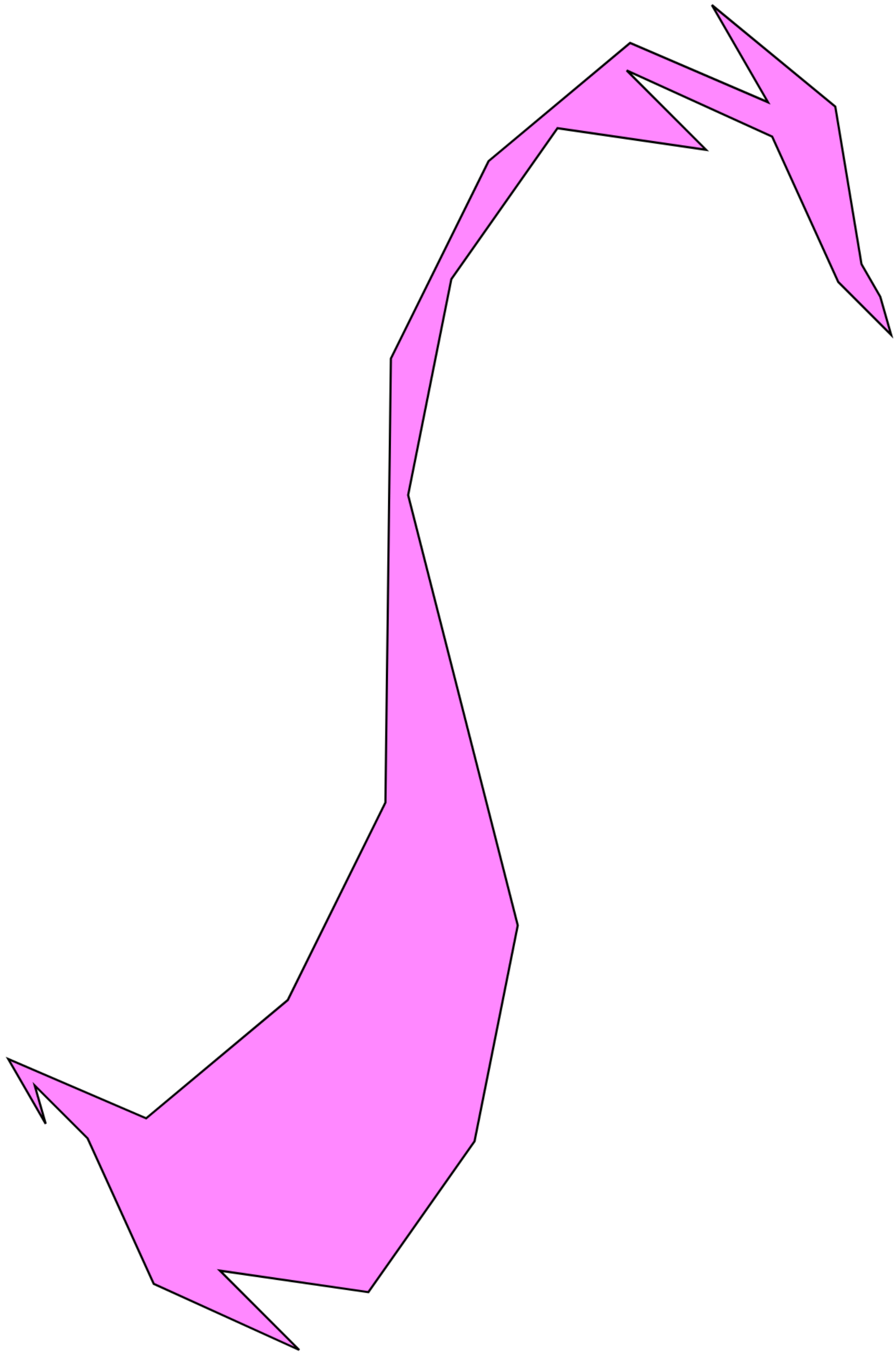

Fig. 3. The new tile with surround number 2, that additionally can tile the plane.

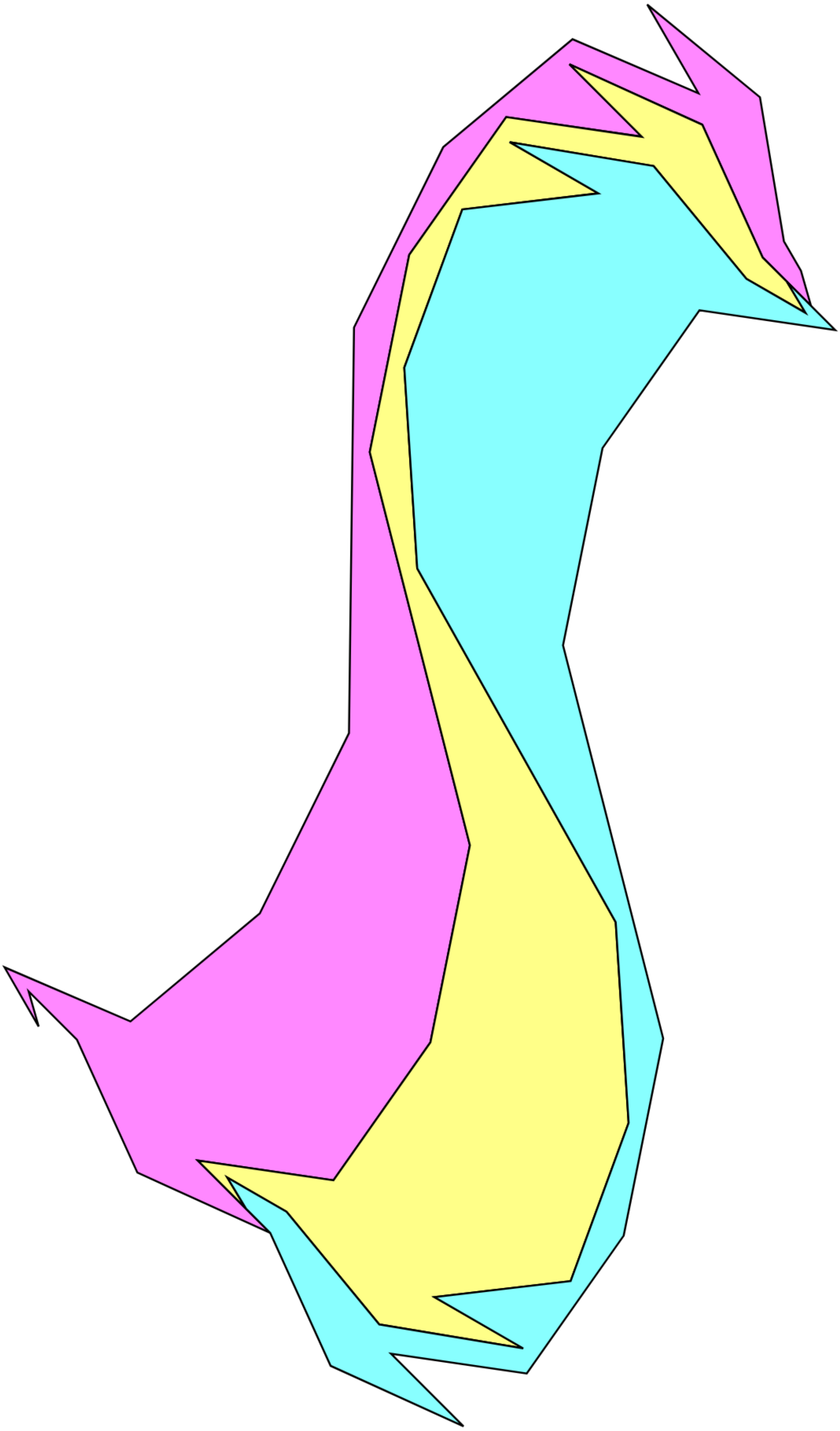

Fig. 4. The tile can be surrounded by two copies of itself. See [5] for SVG graphics.

5 Tiling the plane

The tile and a copy rotated by 180° form a polygon with twofold rotational symmetry (Fig. 5). An infinite stripe made of such tile couples has an upper and a lower boundary that is equal to the boundary of a sequence of isosceles triangles (Fig. 6). Thus, these stripes can be connected in order to fill the whole plane. This tiling is unique and periodic (Fig. 7). If we allow flips of the tile, a different stripe is possible and the two stripes a and b can be combined arbitrarily (Fig. 8). Thus, an infinite number of periodic and non-periodic tilings of the plane are possible. For example: Say c is any finite combination of stripes a and b. Then ...$cccc$... is a periodic tiling and ...$aaacaaa$... with only one c is non-periodic.

Fig. 5 to 8 were created with the app of Arnaud Chéritat [6].

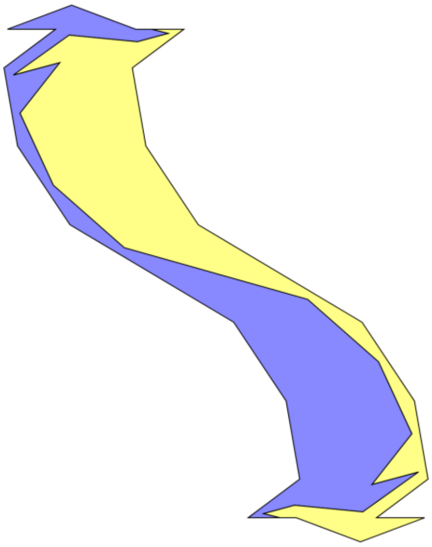

Fig. 5. Symmetric tile couple

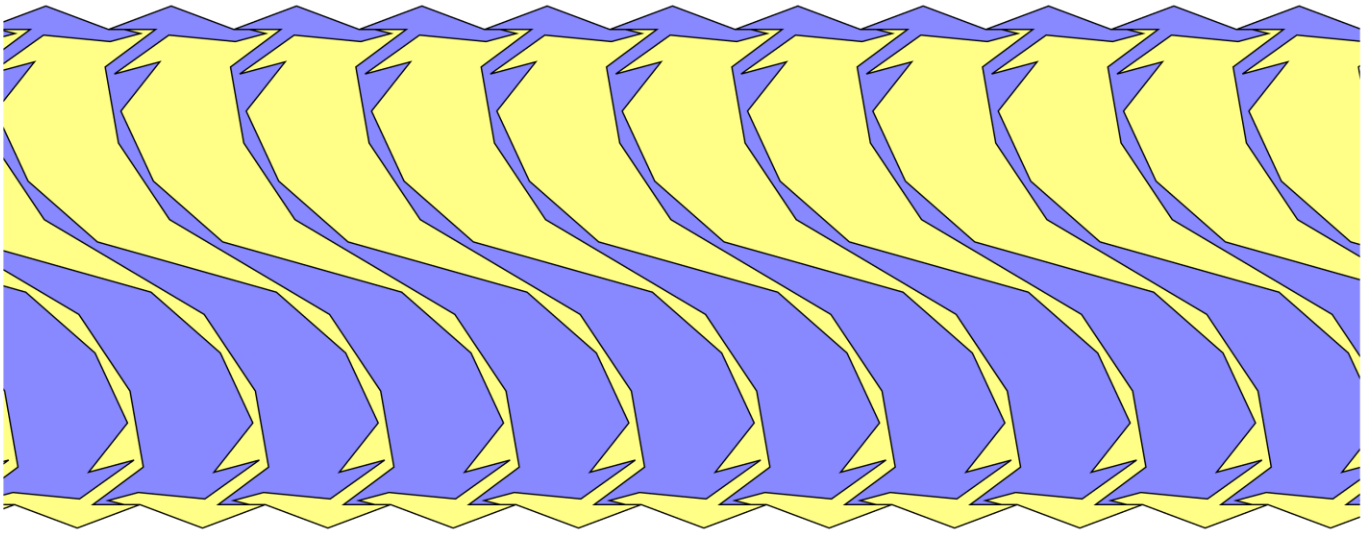

Fig. 6. An infinite stripe of tile couples

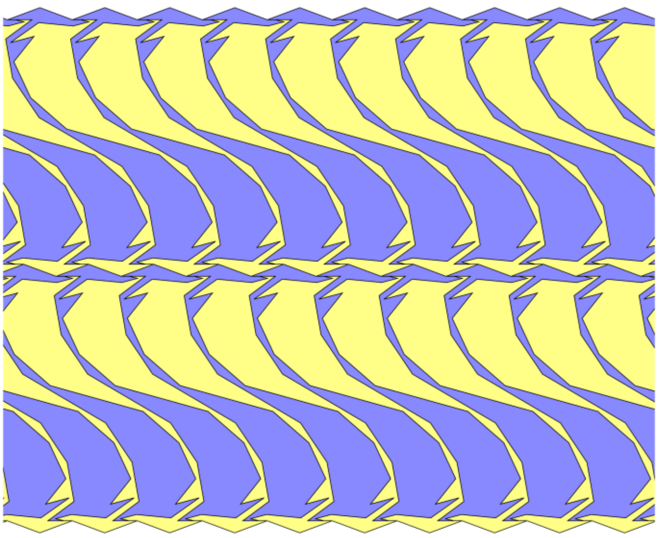

Fig. 7. Tiling the plane without flips

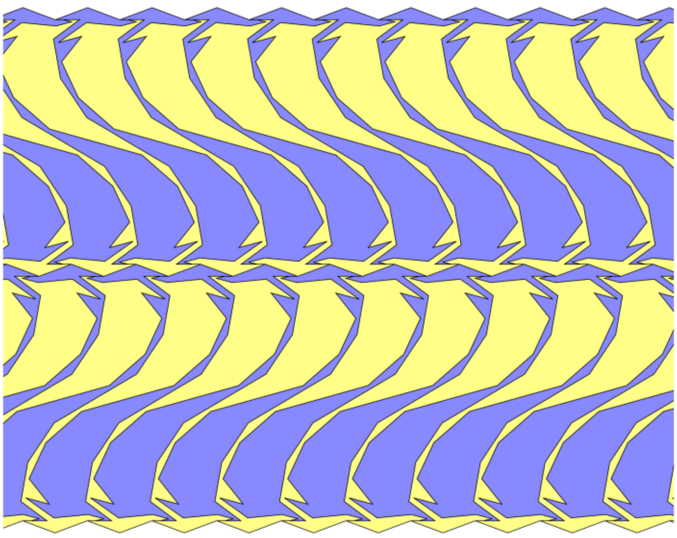

Fig. 8. Tiling the plane including flipped tiles

6 Construction of the new tile

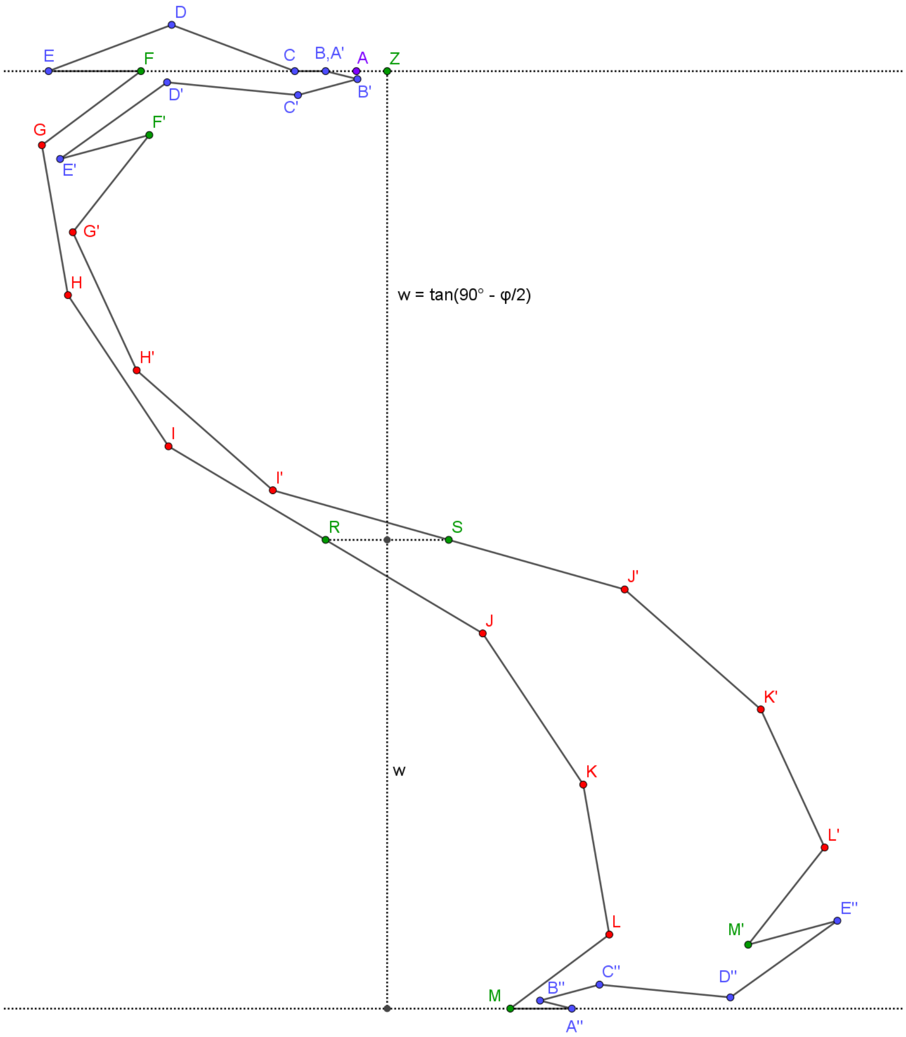

Fig. 9. Positions and names of needed points (drawn with GeoGebra [7])

The construction of the tile can be done for nearly any angle $\varphi < 25°$. If the angle is a divisor of 360°, a disk can be built with a hole in the center. Here $\varphi = 15°$ is chosen.
The triangle RSZ is isosceles with base length 2 (in any unit) and with angle φ at the apex.
The points R, S, Z, and A are not vertices of the tile.

Given are following points: Z(0, 0), R(-1, -w), S(1, -w), A(-0.5, 0), B(-1, 0), C(-1.5, 0), D(-3.5, 0.75), E(-5.5, 0), F(-4, 0), G(-5.608, -1.2), H(-5.185, -3.63), I(-3.552, -6.08)
Underlined coordinates are not fix and can be changed slightly,
but it is important that $x_D = x_C + 2$ and $x_E = x_C + 4$.

refl(P, T) is defined as the point obtained when the point P is reflected at the point T.
rot(P, T) is defined as the point obtained when the point P is rotated around T by the angle φ counterclockwise.

Determine the other points as follows:
M = refl(F, R), L = refl(G, R), K = refl(H, R), and J = refl(I, R).
B' = rot(A, Z),   A' = B   (!)
P' = rot(P, Z) for P = C, D, E, ..., M
P'' = refl(P', S) for P = A, B, C, D, E

In order to achieve a 25-gon leave away the points I, J, I', J'
and replace the coordinates of the point H by (-4.55, -4.777).

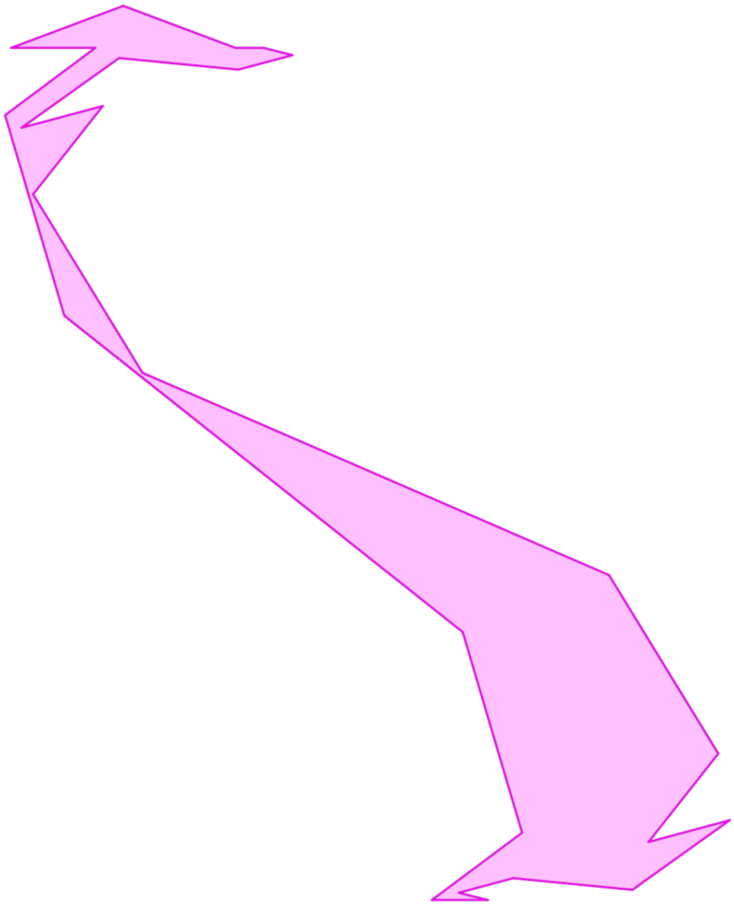

Fig. 10. A 25-gon as a thin version of the original tile

Acknowledgements

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References

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